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Soft rough sets based on covering and their applications

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Abstract

Soft rough based on covering $\mathcal{S}RC$ theory has become a useful and well-known area of research in theories of uncertainty. The present work follows up on this flourishing research topic. We introduce a new model of $\mathcal{S}RC$ in a fusion of soft set theory $\mathcal{S}ST$ and rough set depending on covering $\mathcal{C}RS$. We put forth a definition of soft rough covering approximation space $\mathcal{S}CAS$ via neighborhood concept. Some axiomatic systems of our model of $\mathcal{S}RC$ are discussed. We study the relationship between our model of $\mathcal{S}RC$ and three other $\mathcal{S}RC$-models. An algorithm for reduction of the attributes of the information systems using $\mathcal{S}CAS$ is established. Based on the theoretical discussion, we set forth some applications of our model which will be helpful in decision making process via $\mathcal{S}RC$ theory.

Mathematics Subject Classification: 54A40; 03E72; 54C08

Keywords: Minimal soft neighborhood; Soft rough; Covering; Reduction

1 Introduction

Probability theory, fuzzy set theory $\mathcal{F}ST$ and rough set theory $\mathcal{R}ST$ are familiar methodologies for addressing ambiguity and uncertainty. Pawlak [1] established $\mathcal{R}ST$ as a advantageous technique to treat with inexact and mysterious issues. Currently, $\mathcal{R}ST$ has attracted researchers in various fields such as: information processing, knowledge discovery, data analysis, control and pattern recognition [2, 3]. First, $\mathcal{R}ST$ was built on the equivalence relation for the granulation of the universe. Second, many researchers generalized $\mathcal{R}ST$ by generalizing the equivalence relation. Covering-based rough set models $\mathcal{C}RS$-models is one of the significant generalizations of Pawlak's rough sets. $\mathcal{C}RS$ is more realistic technique that help the researcher to investigate fuzziness and ambiguous of the issues.

Molodtsov [4] has developed a new method for studying fuzziness and vagueness called soft sets theory $\mathcal{S}ST$. Although $\mathcal{S}ST$ embeds fuzzy sets theory $\mathcal{F}ST$, it differs from it. Moreover, $\mathcal{S}ST$ is different from $\mathcal{R}ST$ and the other theories of uncertainty. For example,we require a huge number of experiments to test the stability of the system in probability theory. The lack of resources for parametrization [4] may be the reason for the difficulties associated with these theories of vagueness. $\mathcal{S}ST$ is not subject to the difficulties mentioned above because it has sufficient parameters. Although $\mathcal{S}ST$ deals with fuzziness and vagueness, it has sufficient techniques for parameterizations. The previous advantages of $\mathcal{SST}$ make it popular among professionals and researchers working in various fields. Some theoretical studies Some theoretical studies on $\mathcal{SST}$ and its applications can be found in [5–14].

Feng et al. [15] put forth a likely combination of $\mathcal{RST}$ and $\mathcal{SST}$, although $\mathcal{RST}$ and $\mathcal{SST}$ have distinct approaches to fuzziness. The authors proposed soft rough set concept by parametrization the subsets instead of using the equivalence classes to obtain the lower and upper approximations of the subsets. A connection between N-soft sets and rough structures of various kinds is given by Alcantud et al. [16]. Many researchers studied $\mathcal{SST}$ from different approaches such as soft-type algebraic structures and soft topological structures [17–20].

Soft rough covering set theory $\mathcal{SRC}$ has various applications in different industries. Some of the applications is decision support systems which can be widely used in many types of industrial decision making on different levels [21]. $\mathcal{SRC}$ can be used in new materials design and investigating of material properties [22]. $\mathcal{SRC}$ can be used in technical diagnosis of mechanical things via vibroacoustics manifestation [23]. $\mathcal{SRC}$ is related to neural networks which has many interesting applications in intelligent control for industrial processes [24].

Three types of different soft rough covering models $\mathcal{SRC}$-models are put forth by Li et al. [25] and Yuksel et al. [26, 27]. Their models are considered a combination of $\mathcal{SST}$-models and $\mathcal{CRS}$-models. They discussed an important properties of $\mathcal{SRC}$-models. Interest in $\mathcal{SRC}$ was sparked by these studies and $\mathcal{SRC}$ has turned into a significant and beneficial area of research in fuzziness. Zhan et al. [28, 29] introduced two models of multigranulation rough fuzzy sets and three classes of intuitionistic fuzzy models based on covering while covering based multigranulation fuzzy rough set types is introduced [30] using fuzzy neighborhoods. Zhang et al. [31] generalized fuzzy rough sets by coimplication operators (R-coimplicators and T-coimplicator). In our paper, we set forth $\mathcal{SST}$-models based on $\mathcal{CRS}$ via neighborhood concept.

This paper follows the study of $\mathcal{SRC}$ theory. We introduce in Sect. 2 some notions from Pawlak's $\mathcal{RST}$, $\mathcal{SST}$ and give three models of soft rough set depending on covering as well. Through Sect. 3, we set forth a new model of soft rough based covering. Decision making via soft rough depending on covering is presented in Sect. 4.

2 Basic terminology and results

Soft rough sets based on covering $\mathcal{SRC}$ model is a combination between soft sets theory and rough sets theory depending on covering. Yuksel et al. [27] and Li et al. [25] have presented three models of $\mathcal{SRC}$ and studied their properties. Through this section, we give basic terminology of $\mathcal{RST}$ and $\mathcal{SST}$. We review the definitions of these kinds of $\mathcal{SRC}$-models and discuss their properties as well. Throughout this paper $\mathcal{U}$ denotes a finite nonempty set. By $\mathcal{X}^c$ we mean the complement set of $\mathcal{X}$ in $\mathcal{U}$.

Throughout this section, consider $\mathcal{R}$ is a relation of equivalence on $\mathcal{U}$. Hence, $\mathcal{U}/\mathcal{R} = \{\mathcal{Y}_2, \mathcal{Y}_1, \mathcal{Y}_3, \dots, \mathcal{Y}_m\}$ is a partition on $\mathcal{U}$, where $\mathcal{R}$ is a relation of equivalence which generates the classes of equivalence $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \dots, \mathcal{Y}_m$. With respect to soft set, consider $\mathcal{U}$ is a universe set, $\mathcal{A}$ is a set of parameters on $\mathcal{U}$, $\mathcal{P}(\mathcal{U})$ is the power set of $\mathcal{U}$.

Definition 2.1 [32] Assume that $\mathcal{R}$ is a relation of equivalence on a nonempty set $\mathcal{U}$. For any $\mathcal{X}_1 \subseteq \mathcal{U}$, the set $\underline{\mathcal{R}}(\mathcal{X}_1) = \bigcup\{\mathcal{Y}_i \in \mathcal{U}/\mathcal{R} : \mathcal{Y}_i \subseteq \mathcal{X}_1\}$ is called the lower approximation

(L_{approx}) of $\mathcal{X}_1$ and the set $\overline{\mathcal{R}}(\mathcal{X}_1) = \bigcup\{\mathcal{Y}_i \in \mathcal{U}/\mathcal{R} : \mathcal{Y}_i \cap \mathcal{X}_1 \neq \emptyset\}$ is called the upper approximation ($\mathcal{U}_{\text{approx}}$) of $\mathcal{X}_1$.

Theorem 2.1 [32] *Let* $\mathcal{K} = (\mathcal{U}, \mathcal{R})$ *be an approximation structure and* $\mathcal{X}_1, \mathcal{X}_2 \subseteq \mathcal{U}$. *Hence*:

(1) $\underline{\mathcal{R}}(\mathcal{U}) = \overline{\mathcal{R}}(\mathcal{U}) = \mathcal{U}$;
(2) $\underline{\mathcal{R}}(\emptyset) = \overline{\mathcal{R}}(\emptyset) = \emptyset$;
(3) $\underline{\mathcal{R}}(\mathcal{X}_1) \subseteq \mathcal{X}_1 \subseteq \overline{\mathcal{R}}(X_1)$;
(4) $\mathcal{X}_1 \subseteq \mathcal{X}_2 \Rightarrow \underline{\mathcal{R}}(\mathcal{X}_1) \subseteq \underline{\mathcal{R}}(\mathcal{X}_2)$ *and* $(\overline{\mathcal{R}}(\mathcal{X}_1) \subseteq \overline{\mathcal{R}}(\mathcal{X}_2))$;
(5L) $\underline{\mathcal{R}}(\mathcal{X}_1 \cap \mathcal{X}_2) = \underline{\mathcal{R}}(\mathcal{X}_1) \cap \underline{\mathcal{R}}(\mathcal{X}_2)$ *and* $\underline{\underline{\mathcal{R}}}(\mathcal{X}_1) \cup \underline{\mathcal{R}}(X_2) \subseteq \underline{\mathcal{R}}(\mathcal{X}_1 \cup \mathcal{X}_2)$;
(5H) $\overline{\mathcal{R}}(\mathcal{X}_1 \cap \mathcal{X}_2) \subseteq \overline{\mathcal{R}}(\mathcal{X}) \cap \overline{\mathcal{R}}(\mathcal{X}_2)$ *and* $\hat{\overline{\mathcal{R}}}(\mathcal{X}_1) \cup \overline{\mathcal{R}}(\mathcal{X}_2) = \overline{\mathcal{R}}(\mathcal{X}_1 \cup \mathcal{X}_2)$;
(6) $\underline{\mathcal{R}}(\mathcal{X}_1^c) = (\overline{\mathcal{R}}(\mathcal{X}_1))^c$ *and* $\overline{\mathcal{R}}(\mathcal{X}_1^c) = (\underline{\mathcal{R}}(\mathcal{X}_1))^c$, *where* $(\mathcal{X}_1^c)$ *is the complement of* $\mathcal{X}_1$;
(7) $\underline{\mathcal{R}}(\mathcal{X}_1) = \underline{\mathcal{R}}(\underline{\mathcal{R}}(\mathcal{X}_1))$ *and* $\overline{\mathcal{R}}(\mathcal{X}_1) = \overline{\mathcal{R}}(\overline{\mathcal{R}}(\mathcal{X}_1))$.

Definition 2.2 [33], [34] Consider $\mathcal{C}$ is a family of subsets of the universe $\mathcal{U}$. We call $\mathcal{C}$ a covering of $\mathcal{U}$ if $\cup\mathcal{C} = \mathcal{U}$ where none subset in $\mathcal{C}$ is empty.

Definition 2.3 [33], [34] Suppose that $\mathcal{C}$ is a covering of the non-empty set $\mathcal{U}$. Then, the structure $(\mathcal{U}, \mathcal{C})$ is called rough approximation space depending on covering ($\mathcal{RASC}$).

Definition 2.4 [4] Consider the mapping $\mathcal{F} : \mathcal{A} \to \mathcal{P}(U)$. The structure $\mathcal{G} = (\mathcal{F}, \mathcal{A})$ is called soft set on $\mathcal{U}$. The soft set is a full soft set if $\bigcup_{e_1 \in \mathcal{A}} \mathcal{F}(e_1) = \mathcal{U}$.

Definition 2.5 [35], [36] We say that $\mathcal{S} = (\mathcal{U}, \mathcal{G})$ is a soft rough covering approximation space $\mathcal{S}CAS$, where we fix a soft set $\mathcal{G} = (\mathcal{F}, \mathcal{A})$ on the universe set $\mathcal{U}$.

For $\mathcal{B} \subseteq \mathcal{U}$, we have the following two operators:

$L_{approx}(\mathcal{B}) = \{u_1 \in \mathcal{U} : \exists e_1 \in \mathcal{A} S.T. u_1 \in \mathcal{F}(e_1) \subseteq \mathcal{B}\}$

$\mathcal{U}_{approx}(\mathcal{B}) = \{u_1 \in \mathcal{U} : \exists e_1 \in \mathcal{A} S.T. u_1 \in \mathcal{F}(e_1), \mathcal{F}(e_1) \cap \mathcal{B} \neq \emptyset\}$.

Two operators are called the soft $\mathcal{P}$-lower and $\mathcal{P}$-upper approximation of $\mathcal{B}$, respectively. $\mathcal{B}$ is called soft-definable if $L_{approx}(\mathcal{B}) = \mathcal{U}_{approx}(\mathcal{B})$, if else it is Feng-soft rough(Feng-$\mathcal{S}R$-set).

Definition 2.6 [36], [27] Suppose that $\mathcal{S} = (\mathcal{U}, \mathcal{G})$ is a soft rough covering approximation space ($\mathcal{S}CAS$). For each $y_1 \in \mathcal{U}$, the mapping $\zeta : \mathcal{U} \to \mathcal{P}(\mathcal{A})$ is defined by $\zeta(y_1) = \{e_1 \in \mathcal{A} | y_1 \in \{(e_1)\}$.

For $\mathcal{B} \subseteq \mathcal{U}$, we have the following two operators:

$L_{approx}(\mathcal{B}_\zeta) = \{x_1 \in \mathcal{U} | \zeta(x_2) \neq \zeta(x_1)\}$ for each $x_2 \in \mathcal{B}$

$\mathcal{U}_{approx}(\mathcal{B}_\zeta) = \{x_1 \in \mathcal{U} | \zeta(x_2) = \zeta(x_1)\}$ for some $x_2 \in \mathcal{B}$ are called lower and upper $\mathcal{MSR}$-approximation of $\mathcal{B}$, respectively. $\mathcal{B}$ is called MS-definable if $L_{approx}(\mathcal{B}_\zeta) = \mathcal{U}_{approx}(\mathcal{B}_\zeta)$, otherwise it is an $\mathcal{MSR}$-set.

Definition 2.7 Let $\mathcal{S} = (\mathcal{U}, \mathcal{G})$ be a soft rough covering approximation space $\mathcal{S}CAS$. Consider $\mathcal{X}_1 \subseteq \mathcal{U}$, hence $\underline{\mathcal{F}S}(\mathcal{X}_1) = \bigcup\{\mathcal{F}(e_1) : e_1 \in \mathcal{A}, \mathcal{F}(e_1) \subseteq \mathcal{X}_1\}$ is called the soft lower approximation based on covering of $\mathcal{X}_1$ and the soft upper approximation based on covering of $\mathcal{X}_1$ is defined as $\overline{\mathcal{F}S}(\mathcal{X}_1) = \bigcup\{\mathcal{F}(e_1) : e_1 \in \mathcal{A}, \mathcal{X}_1 \cap \mathcal{F}(e_1) \neq \emptyset\}$. The operator $\overline{\mathcal{R}S}(X_1)$ is defined as $\overline{\mathcal{R}S}(X_1) = \underline{\mathcal{F}S}(\mathcal{X}_1) \cup \{\mathcal{F}(e_1) : \mathcal{F}(e_1) \cap (X_1 - \underline{\mathcal{F}S}(\mathcal{X}_1)) \neq \emptyset, \forall e_1 \in \mathcal{A}\}$.

The operator $\underline{\mathcal{F}S}(\mathcal{X}_1)$ is called Li-soft lower approximation based on covering and the operator $\overline{\mathcal{F}S}(\mathcal{X}_1)$ is called Li-soft upper approximation based on covering. If $\underline{\mathcal{F}S}(\mathcal{X}_1) = \overline{\mathcal{F}S}(\mathcal{X}_1)$, then the set $\mathcal{X}_1$ is Li-$\mathcal{S}RC$ definable, if else $\mathcal{X}_1$ is Li-$\mathcal{S}RC$.

Theorem 2.2 [37] *Let $\mathcal{S} = (\mathcal{U}, \mathcal{G})$ be a $\mathcal{S}CAS$ and $\mathcal{X}_1, \mathcal{X}_2 \subseteq \mathcal{U}$. Hence*:

(1) $\underline{\mathcal{F}S}(\mathcal{U}) = \overline{\mathcal{F}S}(\mathcal{U}) = \mathcal{U}$;

(2) $\underline{\mathcal{F}S} = \overline{\mathcal{F}S}(\emptyset) = \emptyset$;

(3) $\underline{\mathcal{F}S}(\mathcal{X}_1) \subseteq \mathcal{X}_1 \subseteq \overline{\mathcal{F}S}(\mathcal{X}_1)$;

(4) $\mathcal{X}_1 \subseteq \mathcal{X}_2 \Rightarrow (\underline{\mathcal{F}S}(\mathcal{X}_1) \subseteq \underline{\mathcal{F}S}(\mathcal{X}_2))$ *and* $(\overline{\mathcal{F}S}(\mathcal{X}_1) \subseteq \overline{\mathcal{F}S}(\mathcal{X}_2))$;

(5L) $\underline{\mathcal{F}S}(\mathcal{X}_1 \cap \mathcal{X}_2) \subseteq \underline{\mathcal{F}S}(\mathcal{X}_1) \cap \underline{\mathcal{F}S}(\mathcal{X}_2)$ *and* $\underline{\mathcal{F}S}(\mathcal{X}_1) \cup \underline{\mathcal{F}S}(\mathcal{X}_2) \subseteq \underline{\mathcal{F}S}(\mathcal{X}_1 \cup \mathcal{X}_2)$;

(5H) $\overline{\mathcal{F}S}(\mathcal{X}_1 \cap \mathcal{X}_2) \subseteq \overline{\mathcal{F}S}(\mathcal{X}_1) \cap \overline{\mathcal{F}S}(\mathcal{X}_2)$ *and* $\overline{\mathcal{F}S}(\mathcal{X}_1) \cup \overline{\mathcal{F}S}(\mathcal{X}_2) = \overline{\mathcal{F}S}(\mathcal{X}_1 \cup \mathcal{X}_2)$;

(6) $[\underline{\mathcal{F}S}(\mathcal{X}_1)]^c \subseteq \overline{\mathcal{F}S}(\mathcal{X}_1^c)$ *and* $[\overline{\mathcal{F}S}(\mathcal{X}_1)]^c \subseteq \underline{\mathcal{F}S}(\mathcal{X}_1^c)$;

(7) $\underline{\mathcal{F}S}(\underline{\mathcal{F}S}(\mathcal{X}_1)) = \underline{\mathcal{F}S}(\mathcal{X}_1)$ *and* $\overline{\mathcal{F}S}(\mathcal{X}_1) \subseteq \overline{\mathcal{F}S}(\overline{\mathcal{F}S}(\mathcal{X}_1))$;

(8) $\underline{\mathcal{F}S}(\mathcal{F}(e_1)) = \mathcal{F}(e_1)$, *for each* $e_1 \in \mathcal{A}$.

Definition 2.8 [26] Consider $\mathcal{S} = (\mathcal{U}, \mathcal{G})$ is a $\mathcal{S}CAS$, $\mathcal{X}_1 \subseteq \mathcal{U}$. For $x \in \mathcal{U}$, the two operators $\underline{\mathcal{S}S}(\mathcal{X}_1) = \underline{\mathcal{F}S}(\mathcal{X}_1)$, $\overline{\mathcal{S}S}(\mathcal{X}_1) = \underline{\mathcal{F}S}(\mathcal{X}_1) \bigcup \{Md_s(x) : x \in \mathcal{X}_1 - \underline{\mathcal{F}S}(\mathcal{X}_1)\}$ are called Yul-soft lower approximation based on covering and Yul-soft upper approximation based on covering, respectively. The set $\mathcal{X}_1$ is called Yul-$\mathcal{S}RC$ definable if $\underline{\mathcal{S}S}(\mathcal{X}_1) = \overline{\mathcal{S}S}(\mathcal{X}_1)$,otherwise it is Yul-$\mathcal{S}RC$.

Definition 2.9 [27] Consider the structure $\mathcal{S} = (U, \mathcal{G})$ represents a $\mathcal{S}CAS$, $\mathcal{X}_1 \subseteq U$. $\forall x_1 \in U$:the two operators $\underline{\mathcal{T}S}(\mathcal{X}_1) = \underline{\mathcal{F}S}(\mathcal{X}_1)$, $\overline{\mathcal{T}S}(\mathcal{X}_1) = \bigcup\{Md_s(x) : x \in \mathcal{X}\}$ are called Yul et. al.'s second $\mathcal{S}RC$-model. The set $\mathcal{X}_1$ is called Yul2-$\mathcal{S}RC$ definable if $\underline{\mathcal{T}S}(\mathcal{X}_1) = \overline{\mathcal{T}S}(\mathcal{X}_1)$, otherwise it is Yul2-$\mathcal{S}RC$.

3 A new model of soft rough depending on covering

Throughout this section, we put forth a new model of soft rough depending on covering via the concept of soft minimal neighborhood. The properties of the new model and some illustrative examples are given. We discuss the relationship between different kinds of soft rough based on covering $\mathcal{S}RC$. Four different types of topologies are derived from different models of $\mathcal{S}RC$.

Definition 3.1 Let $\mathcal{S} = (\mathcal{U}, \mathcal{G})$ be a $\mathcal{S}CAS$, $\mathcal{U} = \{x_1, x_2, x_3, \ldots, x_n\}$ and $\mathcal{A} = \{e_1, e_2, e_3, \ldots, e_m\}$. Then $\mathcal{M}N_S(x_i) = \bigcap\{\mathcal{F}(e_j) : x_i \in \mathcal{F}(e_j), \forall x_i \in \mathcal{U}\}$, $i = 1, 2, 3, \ldots, n$ and $j = 1, 2, 3, \ldots, m$ is called minimal soft neighborhood of x_i.

Definition 3.2 Suppose that $\mathcal{S} = (\mathcal{U}, \mathcal{G})$ is a $\mathcal{S}CAS$, $x_1 \in \mathcal{U}$. For any $\mathcal{X}_1 \subseteq \mathcal{U}$, the soft covering lower approximation and the soft covering upper approximation are defined as follow, respectively:

$$\underline{\mathcal{M}S}(\mathcal{X}_1) = \{x_1 \in \mathcal{U} : \mathcal{M}N_S(x_1) \subseteq \mathcal{X}_1\},$$
$$\overline{\mathcal{M}S}(\mathcal{X}_1) = \{x_1 \in \mathcal{U} : \mathcal{M}N_S(x_1) \cap \mathcal{X}_1 \neq \emptyset\}.$$

Example 3.1 Let $(U, \mathcal{G})$ be a $\mathcal{S}CAS$ where $G = (\mathcal{F}, \mathcal{A})$ is soft set define in Table 1.

From Table 1, the minimal soft neighborhood is calculated as the following: $\mathcal{M}N_S(x_1) = \{x_1, x_2\}$, $\mathcal{M}N_S(x_2) = \{x_1, x_2\}$, $\mathcal{M}N_S(x_3) = \{x_3\}$, $\mathcal{M}N_S(x_4) = \{x_4, x_5\}$, $\mathcal{M}N_S(x_5) = \{x_5\}$, $\mathcal{M}N_S(x_6) = \{x_3, x_5, x_6\}$. Let $\mathcal{X}_1 = \{x_3, x_4, x_6\}$, so $\underline{\mathcal{M}S}(\mathcal{X}_1) = \{x_3, x_5\}$ and $\overline{\mathcal{M}S}(\mathcal{X}_1) = \{x_3, x_4, x_5, x_6\}$.

Table 1 $G = (\mathcal{F}, \mathcal{A})$

$\hat{\mathcal{U}}$	a_1	a_2	a_3	a_4	a_5
x_1	1	1	1	0	0
x_2	1	1	1	0	0
x_3	0	1	0	1	1
x_4	0	0	1	1	0
x_5	0	0	1	1	1
x_6	0	0	0	1	1

Theorem 3.1 *Let $\mathcal{S} = (\mathcal{U}, \mathcal{G})$ be a $\mathcal{S}CAS$ and $\mathcal{X}_1, \mathcal{X}_2 \subseteq \mathcal{U}$. Hence*:

(1) $\underline{\mathcal{M}S}(\mathcal{U}) = \overline{\mathcal{M}S}(\mathcal{U}) = \mathcal{U}$;
(2) $\underline{\mathcal{M}S}(\emptyset) = \overline{\mathcal{M}S}(\emptyset) = \emptyset$;
(3) $\underline{\mathcal{M}S}(\mathcal{X}_1) \subseteq \mathcal{X}_1 \subseteq \overline{\mathcal{M}S}(\mathcal{X}_1)$;
(4) $\mathcal{X}_1 \subseteq \mathcal{X}_2 \Rightarrow (\underline{\mathcal{M}S}(\mathcal{X}_1) \subseteq \underline{\mathcal{M}S}(\mathcal{X}_2))$ *and* $(\overline{\mathcal{M}S}(\mathcal{X}_1) \subseteq \overline{\mathcal{M}S}(\mathcal{X}_2))$;
(5) $\underline{\mathcal{M}S}(\mathcal{X}_1 \cap \mathcal{X}_2) = \underline{\mathcal{M}S}(\mathcal{X}_1) \cap \underline{\mathcal{M}S}(\mathcal{X}_2)$ *and* $\underline{\mathcal{M}S}(\mathcal{X}_1 \cup \mathcal{X}_2) = \underline{\mathcal{M}S}(\mathcal{X}_1) \cup \underline{\mathcal{M}S}(\mathcal{X}_2)$;
(6) $(\underline{\mathcal{M}S}(\mathcal{X}_1))^c = \overline{\mathcal{M}S}(\mathcal{X}_1^c)$ *and* $(\overline{\mathcal{M}S}(\mathcal{X}_1))^c = \underline{\mathcal{M}S}(\mathcal{X}_1^c)$;
(7) $\underline{\mathcal{M}S}(\underline{\mathcal{M}S}(\mathcal{X}_1)) = \underline{\mathcal{M}S}(\mathcal{X}_1)$ *and* $\overline{\mathcal{M}S}(\overline{\mathcal{M}S}(\mathcal{X}_1)) = \overline{\mathcal{M}S}(\mathcal{X}_1)$;
(8) $\underline{\mathcal{M}S}(\mathcal{F}(e_1)) = \mathcal{F}(e_1), \forall e_1 \in \mathcal{A}$.

Proof It is obvious to prove parts 1 and 2. The proof of other parts is discussed in the following:

(3) Take $x_1 \in \underline{\mathcal{M}S}(\mathcal{X}_1)$, then $x_1 \in \mathcal{M}N_S(x_1)$ and $\mathcal{M}N_S(x_1) \subseteq \mathcal{X}_1$. Hence $x_1 \in \mathcal{X}_1$ and $\underline{\mathcal{M}S}(\mathcal{X}_1) \subseteq \mathcal{X}_1$. Also, Select $x_1 \in \mathcal{X}_1$, $x_1 \in \mathcal{M}N_S(x_1)$, then $\mathcal{M}N_S(x_1) \cap \mathcal{X}_1 \neq \emptyset$. Therefore, $x_1 \in \overline{\mathcal{M}S}(\mathcal{X}_1)$ and $\mathcal{X}_1 \subseteq \overline{\mathcal{M}S}(\mathcal{X}_1)$
(4) Take $x_1 \in \underline{\mathcal{M}S}(\mathcal{X}_1)$, $x_1 \in \mathcal{M}N_S(x_1) \subseteq \mathcal{X}_1 \subseteq \mathcal{X}_2$. Then, $x_1 \in \underline{\mathcal{M}S}(\mathcal{X}_2)$ and $\underline{\mathcal{M}S}(\mathcal{X}_1) \subseteq \underline{\mathcal{M}S}(\mathcal{X}_2)$. Similarly, $\overline{\mathcal{M}S}(\mathcal{X}_1) \subseteq \overline{\mathcal{M}S}(\mathcal{X}_2))$.
(5) $\underline{\mathcal{M}S}(\mathcal{X}_1 \cap \mathcal{X}_2) = \{x_1 \in \mathcal{U} : \mathcal{M}N_S(x_1) \subseteq (\mathcal{X}_1 \cap \mathcal{X}_2)\} = \{x_1 \in \mathcal{U} : \mathcal{M}N_S(x_1) \subseteq \mathcal{X}_1\}$ and $\{x_1 \in \mathcal{U} : \mathcal{M}N_S(x_1) \subseteq \mathcal{X}_2\} = \{x_1 \in \mathcal{U} : \mathcal{M}N_S(x_1) \subseteq \mathcal{X}_1\} \cap \{x_1 \in \mathcal{U} : \mathcal{M}N_S(x_1) \subseteq \mathcal{X}_2\} = \underline{\mathcal{M}S}(\mathcal{X}_1) \cap \underline{\mathcal{M}S}(\mathcal{X}_2)$. Similarly, $\underline{\mathcal{M}S}(\mathcal{X}_1 \cup \mathcal{X}_2) = \underline{\mathcal{M}S}(\mathcal{X}_1) \cup \underline{\mathcal{M}S}(\mathcal{X}_2)$
(6) Select $x_1 \notin \overline{\mathcal{M}S}(\mathcal{X}_1^c) \Leftrightarrow \{x_1 \in \mathcal{U} : \mathcal{M}N_S(x_1) \cap \mathcal{X}_1^c = \emptyset\} \Leftrightarrow \{x_1 \in \mathcal{U} : \mathcal{M}N_S(x_1) \subseteq \mathcal{X}_1\} \Leftrightarrow x_1 \in \underline{\mathcal{M}S}(\mathcal{X}_1) \Leftrightarrow x_1 \notin [\underline{\mathcal{M}S}(\mathcal{X}_1)]^c$. Therefore, $\overline{\mathcal{M}S}(\mathcal{X}_1^c) = [\underline{\mathcal{M}S}(\mathcal{X}_1)]^c$. Similarly, $(\overline{\mathcal{M}S}(\mathcal{X}_1))^c = \underline{\mathcal{M}S}(\mathcal{X}_1^c)$.
(7) We need to prove $\underline{\mathcal{M}S}(\underline{\mathcal{M}S}(\mathcal{X}_1)) \subseteq \underline{\mathcal{M}S}(\mathcal{X}_1)$ and $\underline{\mathcal{M}S}(\mathcal{X}_1) \subseteq \underline{\mathcal{M}S}(\underline{\mathcal{M}S}(\mathcal{X}_1))$. From (5), the first inclusion is obvious. Secondly, let $x_1 \in \underline{\mathcal{M}S}(\mathcal{X}_1)$,then $\mathcal{M}N_S(x_1) \subseteq \mathcal{X}_1$ and $\underline{\mathcal{M}S}(\mathcal{M}N_S(x_1) \subseteq \underline{\mathcal{M}S}(\mathcal{X}_1)$. Let $x_2 \in \mathcal{M}N_S(x_1)$, then $\mathcal{M}N_S(x_2) \subseteq \mathcal{M}N_S(x_1)$ and $x_2 \in \underline{\mathcal{M}S}(\mathcal{M}N_S(x_1))$. Hence, $\mathcal{M}N_S(x_1) \subseteq \underline{\mathcal{M}S}(\mathcal{M}N_S(x_1)) \subseteq \underline{\mathcal{M}S}(\mathcal{X}_1)$. Thus, $x_1 \in \underline{\mathcal{M}S}(\underline{\mathcal{M}S}(\mathcal{X}_1))$. Therefore, $\underline{\mathcal{M}S}(\mathcal{X}_1) \subseteq \underline{\mathcal{M}S}(\underline{\mathcal{M}S}(\mathcal{X}_1))$. Similarly, $\overline{\mathcal{M}S}(\overline{\mathcal{M}S}(\mathcal{X}_1)) = \overline{\mathcal{M}S}(\mathcal{X}_1)$
(8) Since $\underline{\mathcal{M}S}(\mathcal{F}(e_1)) \subseteq \mathcal{F}(e_1)$. We need to prove $\mathcal{F}(e_1) \subseteq \underline{\mathcal{M}S}(\mathcal{F}(e_1))$. Pick $x_1 \in \mathcal{F}(e_1)$. Thus $x_1 \in \mathcal{M}N_S(x_1) \subseteq \mathcal{F}(e_1)$. Hence $x_1 \in \underline{\mathcal{M}S}(\mathcal{F}(e_1))$. Therefore $\mathcal{F}(e_1) \subseteq \underline{\mathcal{M}S}(\mathcal{F}(e_1))$. □

Remark 3.1 Let $\mathcal{S} = (\mathcal{U}, \mathcal{G})$ be a $\mathcal{S}CAS$ and $\mathcal{X}_1, \mathcal{X}_2 \subseteq \mathcal{U}$. Thus, the following equalities do not hold generally:

(IL) $\underline{\mathcal{M}S}(\mathcal{X}_1) \cup \underline{\mathcal{M}S}(\mathcal{X}_2) = \underline{\mathcal{M}S}(\mathcal{X}_1 \cup \mathcal{X}_2)$;
(IH) $\overline{\mathcal{M}S}(\mathcal{X}_1 \cap \mathcal{X}_2) = \overline{\mathcal{M}S}(\mathcal{X}_1) \cap \overline{\mathcal{M}S}(\mathcal{X}_2)$;

Remark 3.1 is shown throughout the next example.

Example 3.2 Suppose that $\mathcal{S} = (\mathcal{U},\mathcal{G})$ is a $\mathcal{SCAS}$,$A = \{a_1,a_3,a_2,a_5,a_4\}$, $\mathcal{U} = \{x_1,x_3,x_2,x_6,x_5,x_4,x_8,x_7\}$, $\mathcal{F}(a_1) = \{x_1,x_3,x_6\}$, $\mathcal{F}(a_2) = \{x_3,x_2,x_7,x_6\}$, $\mathcal{F}(a_3) = \{x_3,x_2,x_8,x_7\}$, $\mathcal{F}(a_4) = \{x_1,x_4,x_5,x_6\}$, $\mathcal{F}(a_5) = \{x_5,x_4,x_8,x_7\}$. Hence, $MN_S(x_1) = \{x_1,x_6\}$, $MN_S(x_2) = \{x_3,x_2,x_7\}$, $MN_S(x_3) = \{x_3\}$, $MN_S(x_4) = \{x_5,x_4\}$, $MN_S(x_5) = \{x_4,x_5\}$, $MN_S(x_6) = \{x_6\}$, $MN_S(x_7) = \{x_7\}$, $MN_S(x_8) = \{x_7,x_8\}$. Suppose that $\mathcal{X}_1 = \{x_3,x_2,x_1\}$, $\mathcal{X}_2 = \{x_1,x_8,x_7\}$ and $\mathcal{X}_1 \cup \mathcal{X}_2 = \{x_1,x_3,x_2,x_8,x_7\}$. Thus $\underline{\mathcal{MS}}(\mathcal{X}_1) = \{x_3\}$, $\underline{\mathcal{MS}}(\mathcal{X}_2) = \{x_8,x_7\}$, $\underline{\mathcal{MS}}(\mathcal{X}_1) \cup \underline{\mathcal{MS}}(\mathcal{X}_2) = \{x_3,x_8,x_7\}$ and $\underline{\mathcal{MS}}(\mathcal{X}_1 \cup \mathcal{X}_2) = \{x_3,x_2,x_8,x_7\}$. Also, $\mathcal{X}_1 \cap \mathcal{X}_2 = \{x_1\}$, $\overline{\mathcal{MS}}(\mathcal{X}_1) = \{x_1,x_3,x_2,x_6\}$, $\overline{\mathcal{MS}}(\mathcal{X}_2) = \{x_1,x_2,x_7,x_8\}$, $\overline{\mathcal{MS}}(\mathcal{X}_1) \cap \overline{\mathcal{MS}}(\mathcal{X}_2) = \{x_2,x_1\}$ and $\overline{\mathcal{MS}}(\mathcal{X}_1 \cap \mathcal{X}_2) = \{x_1\}$.

The next theorem shows the relationship among different kinds of lower and upper operators.

Theorem 3.2 *Suppose that $\mathcal{S} = (\mathcal{U},\mathcal{G})$ is a $\mathcal{SCAS}$ and $\mathcal{X}_1 \subseteq \mathcal{U}$. Hence: the following axioms are satisfied:*

(i) $\underline{\mathcal{FS}}(\mathcal{X}_1) \subseteq \underline{\mathcal{MS}}(\mathcal{X}_1)$,

(ii) $\overline{\mathcal{MS}}(\mathcal{X}_1) \subseteq \overline{\mathcal{FS}}(\mathcal{X}_1)$,

(iii) $\overline{\mathcal{SS}}(\mathcal{X}_1) \subseteq \overline{\mathcal{TS}}(\mathcal{X}_1) \subseteq \overline{\mathcal{FS}}(\mathcal{X}_1)$

Proof

(i) Take $x \in \underline{\mathcal{FS}}(\mathcal{X}_1)$, then there exists $\mathcal{F}(a_1) \subseteq \mathcal{X}_1$, $a_1 \in \mathcal{A}$ such that $x \in \mathcal{F}(a_1)$. So, $x \in \mathcal{M}N_S(x) \subseteq \mathcal{F}(a_1) \subseteq \mathcal{X}_1$. Therefore, $x \in \underline{\mathcal{MS}}(\mathcal{X}_1)$ and $\underline{\mathcal{FS}}(\mathcal{X}_1) \subseteq \underline{\mathcal{MS}}(\mathcal{X}_1)$.

(ii) Select $x \in \overline{\mathcal{MS}}(\mathcal{X}_1)$, then $x \in \mathcal{M}N_S(x) \cap \mathcal{X}_1 \neq \emptyset$. So, there exists $\mathcal{F}(a_1)$, $a_1 \in \mathcal{A}$ such that $x \in \mathcal{M}N_S(x) \subseteq \mathcal{F}(a_1)$. Thus, $x \in \mathcal{F}(a_1) \cap \mathcal{X}_1 \neq \emptyset$ and $x \in \overline{\mathcal{FS}}(\mathcal{X}_1)$. Hence $\overline{\mathcal{MS}}(\mathcal{X}_1) \subseteq \overline{\mathcal{FS}}(\mathcal{X}_1)$.

(iii) From definitions $\overline{\mathcal{SS}}(\mathcal{X}_1) = \underline{\mathcal{FS}}(\mathcal{X}_1) \bigcup \{\mathcal{M}d_S(x) : x \in \mathcal{X}_1 - \underline{\mathcal{FS}}(\mathcal{X}_1)\}$, $\overline{\mathcal{TS}}(\mathcal{X}_1) = \bigcup \{\mathcal{M}d_S(x) : x \in \mathcal{X}_1\} = \bigcup \{\mathcal{M}d_S(x) : x \in \underline{\mathcal{FS}}(\mathcal{X}_1)\} \bigcup \{\mathcal{M}d_S(x) : x \in \mathcal{X}_1 - \underline{\mathcal{FS}}(\mathcal{X}_1)\}$. This leads to $\overline{\mathcal{SS}}(\mathcal{X}_1) \subseteq \overline{\mathcal{TS}}(\mathcal{X}_1)$. $\overline{\mathcal{TS}}(\mathcal{X}_1) \subseteq \overline{\mathcal{FS}}(\mathcal{X}_1)$ is obvious by the definitions of the operators. □

In general, the equality equation doesn't hold in the above theorem as shown in the next examples.

Example 3.3 According to Example 3.2, consider $\mathcal{X}_1 = \{x_2,x_3,x1\}$. So $\underline{\mathcal{FS}}(\mathcal{X}_1) = \emptyset$, $\underline{\mathcal{MS}}(\mathcal{X}_1) = \{x_2\}$, $\overline{\mathcal{FS}}(\mathcal{X}_1) = \mathcal{U}$ and $\overline{\mathcal{MS}}(\mathcal{X}_1) = \{x_2,x_1,x_6,x_3\}$. Therefore, $\underline{\mathcal{FS}}(\mathcal{X}_1) \subseteq \underline{\mathcal{MS}}(\mathcal{X}_1)$ and $\overline{\mathcal{MS}}(\mathcal{X}_1) \subseteq \overline{\mathcal{FS}}(\mathcal{X}_1)$.

From Example 3.3, we conclude the reverse inclusion doesn't hold in general as $\underline{\mathcal{MS}}(\mathcal{X}_1) \nsubseteq \underline{\mathcal{FS}}(\mathcal{X}_1)$ and $\overline{\mathcal{FS}}(\mathcal{X}_1) \nsubseteq \overline{\mathcal{MS}}(\mathcal{X}_1)$. Therefore, the equality doesn't hold as well.

Example 3.4 Suppose that $\mathcal{S} = (\mathcal{U},\mathcal{G})$ is a $\mathcal{SCAS}$, $\mathcal{U} = \{x_2,x_1,x_4,x_3\}$, $\mathcal{A} = \{a_2,a_1,a_4,a_3,a_5\}$, $\mathcal{F}(a_1) = \{x_1,x_2,x_3\}$, $\mathcal{F}(a_2) = \{x_1,x_3\}$, $\mathcal{F}(a_3) = \{x_2,x_4\}$, $\mathcal{F}(a_4) = \{x_4,x_3\}$, $\mathcal{F}(a_5) = \mathcal{U}$. So, $\mathcal{M}N_S(x_1) = \{x_1,x_2\}$, $\mathcal{M}N_S(x_2) = \{x_2\}$, $\mathcal{M}N_S(x_3) = \{x_1,x_3\}$, $\mathcal{M}N_S(x_4) = \{x_4\}$. Consider $\mathcal{X}_1 = \{x_3,x_1\}$, hence $\overline{\mathcal{FS}}(\mathcal{X}_1) = \mathcal{U}$, $\overline{\mathcal{TS}}(\mathcal{X}_1) = \{x_3,x_1\}$ and $\overline{\mathcal{TS}}(\mathcal{X}_1) \subseteq \overline{\mathcal{FS}}(\mathcal{X}_1)$. Let $\mathcal{X}_2 = \{x_2\}$, so $\overline{\mathcal{SS}}\{x_2\} = \{x_2,x_4\}$, $\overline{\mathcal{TS}}(\mathcal{X}_2) = \hat{\mathcal{U}}$ and $\overline{\mathcal{SS}}(\mathcal{X}_2) \subseteq \overline{\mathcal{TS}}(\mathcal{X}_2)$.

Theorem 3.3 *Suppose that $\mathcal{S} = (\mathcal{U},\mathcal{G})$ is a $\mathcal{SCAS}$, $X_1 \subseteq \mathcal{U}$. $\underline{\mathcal{FS}}(\mathcal{X}_1) = \underline{\mathcal{MS}}(\mathcal{X}_1)$ if and only if $\underline{\mathcal{MS}}(\mathcal{X}_1)$ is union of $\mathcal{F}(a_i)$ for each $a_i \in \mathcal{A}$, $i = 1,2,\ldots,n$.*

Proof Firstly; suppose that $\underline{\mathcal{FS}}(\mathcal{X}_1) = \underline{\mathcal{MS}}(\mathcal{X}_1)$, then $\underline{\mathcal{MS}}(\mathcal{X}_1) = \bigcup\{\mathcal{F}(a_i) : \mathcal{F}(a_i) \subseteq \mathcal{X}_1\}$, $\forall a_i \in \mathcal{A}, i = 1, 2, \ldots, n$. Conversely; suppose that $\underline{\mathcal{MS}}(\mathcal{X}_1)$ is union of $\mathcal{F}(a_i)$ for each $a_i \in \mathcal{A}$, $i = 1, 2, \ldots, n$. So $\underline{\mathcal{MS}}(\mathcal{X}_1) = \mathcal{F}(a_1) \cup \mathcal{F}(a_2) \cup \mathcal{F}(a_3) \cup \cdots \cup \mathcal{F}(a_n)$, $\mathcal{F}(a_i) \subseteq \mathcal{X}_1$. Hence, $\mathcal{F}(a_i) \subseteq \underline{\mathcal{FS}}(\mathcal{X}_1)$ and $\underline{\mathcal{MS}}(\mathcal{X}_1) = \bigcup\{\mathcal{F}(a_i)\} \subseteq \underline{\mathcal{FS}}(\mathcal{X}_1)$. By Theorem 4.2. $\underline{\mathcal{FS}}(\mathcal{X}_1) \subseteq \underline{\mathcal{MS}}(\mathcal{X}_1)$, then $\underline{\mathcal{FS}}(\mathcal{X}_1) = \underline{\mathcal{MS}}(\mathcal{X}_1)$. □

Theorem 3.4 *Let $\mathcal{S} = (\mathcal{U}, \mathcal{G})$ be a $\mathcal{SCAS}$, $\mathcal{X}_1 \subseteq \mathcal{U}$. $\overline{\mathcal{FS}}(\mathcal{X}_1) = \overline{\mathcal{MS}}(\mathcal{X}_1)$ if and only if G forms a partition.*

Proof Consider that G is a partition. Then, $\forall a_1, a_2 \in \mathcal{A}$, $\mathcal{F}(a_1) \cap \mathcal{F}(a_2) = \emptyset$. Hence, $\forall x_1 \in \mathcal{U}$, $x_1 \in \mathcal{F}(a_1)$, $\mathcal{MN}_S(x_1) = \mathcal{F}(a_1)$. So, $\overline{\mathcal{FS}}(\mathcal{X}_1) = \bigcup\{\mathcal{F}(a_1) : \mathcal{X}_1 \cap \mathcal{F}(a_1) \neq \emptyset\} = \{x_1 \in \hat{\mathcal{U}} : \mathcal{MN}_S(x_1) \cap \mathcal{X}_1 \neq \emptyset\} = \overline{\mathcal{MS}}(\mathcal{X}_1)$. Conversely, suppose that $\overline{\mathcal{FS}}(\mathcal{X}_1) = \overline{\mathcal{MS}}(\mathcal{X}_1)$, $\mathcal{F}(a_1) \cap \mathcal{F}(a_2) \neq \emptyset$, $\forall a_1, a_2 \in \mathcal{A}$. Hence, $\overline{\mathcal{RS}}(\mathcal{F}(a_1)) = \mathcal{F}(a_1)$. But $\overline{\mathcal{RS}}(a_1) = \overline{\mathcal{FS}}(a_1)$, so $\overline{\mathcal{FS}}(\mathcal{F}(a_1)) = \mathcal{F}(a_1)$. Since $\mathcal{F}(a_2) \subseteq \overline{\mathcal{FS}}(\mathcal{F}(a_1))$ and $\overline{\mathcal{RS}}(\mathcal{F}(a_2)) = \mathcal{F}(a_2)$. Therefore, $\overline{\mathcal{RS}}(\mathcal{F}(a_2)) = \mathcal{F}(a_2) \nsubseteq \mathcal{F}(a_1) \subseteq \overline{\mathcal{FS}}(\mathcal{F}(a_2))$. This leads to contradiction and hence $\mathcal{F}(a_1) \cap \mathcal{F}(a_2) \neq \emptyset$, G is a partition. □

Definition 3.3 Let $\mathcal{S} = (\mathcal{U}, \mathcal{G})$ be a $\mathcal{SCAS}$, $\mathcal{X}_1 \subseteq \mathcal{U}$. Then the covering soft positive region, the covering soft negative region and the covering soft boundary region of $\mathcal{X}_1$ are defined respectively:

$POS_{SC}(\mathcal{X}_1) = \underline{\mathcal{MS}}(\mathcal{X}_1)$,

$NEG_{SC}(\mathcal{X}_1) = \hat{\mathcal{U}} - \overline{\mathcal{MS}}(\mathcal{X}_1)$,

$BND_{SC}(\mathcal{X}_1) = \overline{\mathcal{MS}}(\mathcal{X}_1) - \underline{\mathcal{MS}}(\mathcal{X}_1)$ It is obvious that if $\overline{\mathcal{MS}}(\mathcal{X}_1) = \underline{\mathcal{MS}}(\mathcal{X}_1)$, then $BND_{SC}(\mathcal{X}_1) = \emptyset$ and $\mathcal{X}_1$ is soft covering exact set. The accuracy measure of the approximation is defined by

$$\eta_{SC}(\mathcal{X}_1) = \frac{|\underline{\mathcal{MS}}(\mathcal{X}_1)|}{|\overline{\mathcal{MS}}(\mathcal{X}_1)|}$$

Example 3.5 Continued Example 3.2, the accuracy measure of the approximation

$$\eta_{SC}(\mathcal{X}_1) = \frac{|\{x_3\}|}{|\{x_1, x_3, x_2, x_6\}|} = \frac{1}{4}.$$

The accuracy measure of the approximation

$$\eta_{SC}(\mathcal{X}_2) = \frac{|\{x_8, x_7\}|}{|\{x_1, x_2, x_7, x_8\}|} = \frac{1}{2}$$

Definition 3.4 Let $\mathcal{S} = (\mathcal{U}, \mathcal{G})$ be a $\mathcal{SCAS}$, $\mathcal{X}_1 \subseteq \mathcal{U}$. Then

(i) $\mathcal{X}_1$ is roughly $\mathcal{SC}$-definable if $\underline{\mathcal{MS}}(\mathcal{X}_1) \neq \emptyset$ and $\overline{\mathcal{MS}}(\mathcal{X}_1) \neq \mathcal{U}$,

(ii) $\mathcal{X}_1$ is internally $\mathcal{SC}$-undefinable if $\underline{\mathcal{MS}}(\mathcal{X}_1) = \emptyset$ and $\overline{\mathcal{MS}}(\mathcal{X}_1) \neq \mathcal{U}$,

(iii) $\mathcal{X}_1$ is externally $\mathcal{SC}$-undefinable if $\underline{\mathcal{MS}}(\mathcal{X}_1) \neq \emptyset$ and $\overline{\mathcal{MS}}(\mathcal{X}_1) = \mathcal{U}$,

(iv) $\mathcal{X}_1$ is totally $\mathcal{SC}$-undefinable if $\underline{\mathcal{MS}}(\mathcal{X}_1) = \emptyset$ and $\overline{\mathcal{MS}}(\mathcal{X}_1) = \mathcal{U}$,

Definition 3.5 Suppose that $\mathcal{S} = (\mathcal{U}, \mathcal{G})$ is a $\mathcal{SCAS}$, $G = (\mathcal{F}, \mathcal{A})$, $\mathcal{X}_1 \subseteq \mathcal{U}$. Then the membership degree based on soft rough of $\mathcal{X}_1$ is defined by:

$$\mu^G_{\mathcal{X}_1}(x_1, \mathcal{X}_1) = \frac{|\mathcal{X}_1 \cap MN_S(x_1)|}{|MN_S(x_1)|}, \quad \forall x_1 \in \mathcal{U}.$$

It is obvious that $\mu^{G}_{\mathcal{X}_1}(x_1,\mathcal{X}_1)$ is a fuzzy set of $\mathcal{U}$ and lies in $[0,1]$, $\forall x_1\in\mathcal{U}$

Example 3.6 Consider Example 4.1, for $\mathcal{X}_1=\{x_1,x_3,x_4,x_6,x_7\}$, then membership degree is $\mu^{G}_{\mathcal{X}_1}(x_1,\mathcal{X}_1)=1$, $\mu^{G}_{\mathcal{X}_1}(x_2,\mathcal{X}_1)=\frac{1}{3}$, $\mu^{G}_{\mathcal{X}_1}(x_3,\mathcal{X}_1)=1$, $\mu^{G}_{\mathcal{X}_1}(x_4,\mathcal{X}_1)=\frac{1}{2}$, $\mu^{G}_{\mathcal{X}_1}(x_5,\mathcal{X}_1)=\frac{1}{2}$, $\mu^{G}_{\mathcal{X}_1}(x_6,\mathcal{X}_1)=1$, $\mu^{G}_{\mathcal{X}_1}(x_7,\mathcal{X}_1)=1$, $\mu^{G}_{\mathcal{X}_1}(x_8,\mathcal{X}_1)=\frac{1}{2}$.

Theorem 3.5 *Let $\mathcal{S}=(\mathcal{U},\mathcal{G})$ be a $\mathcal{S}CAS$, $\mathcal{X}_1\subseteq\mathcal{U}$. Then,*
(i) $\mu^{G}_{\mathcal{X}_1}(x,\mathcal{X}_1)=1\Leftrightarrow x\in POS_{SC}(\mathcal{X}_1)$, $\forall x\in\mathcal{U}$;
(ii) $\mu^{G}_{\mathcal{X}_1}(x,\mathcal{X}_1)=0\Leftrightarrow x\in NEG_{SC}(\mathcal{X}_1)$, $\forall x\in\mathcal{U}$;
(iii) $0<\mu^{G}_{\mathcal{X}_1}(x,\mathcal{X}_1)<1\Leftrightarrow x\in BND_{SC}(\mathcal{X}_1)$, $\forall x\in\mathcal{U}$.

Proof
(i) $\mu^{G}_{\mathcal{X}_1}(x,\mathcal{X}_1)=1\Leftrightarrow MN_S(x)\subseteq\mathcal{X}_1\Leftrightarrow x\in\underline{\mathcal{M}S}(\mathcal{X}_1)\Leftrightarrow x\in POS_{SC}(\mathcal{X}_1)$, $\forall x\in\mathcal{U}$,
(ii) $\mu^{G}_{\mathcal{X}_1}(x,\mathcal{X}_1)=0\Leftrightarrow MN_S(x)\cap\mathcal{X}_1=\emptyset\Leftrightarrow x\notin\overline{\mathcal{M}S}(\mathcal{X}_1)\Leftrightarrow x\in NEG_{SC}(\mathcal{X}_1)$, $\forall x\in\mathcal{U}$,
(iii) $0<\mu^{G}_{\mathcal{X}_1}(x,\mathcal{X}_1)<1\Leftrightarrow x\in\overline{\mathcal{M}S}(\mathcal{X}_1)-\underline{\mathcal{M}S}(\mathcal{X}_1)\Leftrightarrow x\in BND_{SC}(\mathcal{X}_1)$, $\forall x\in\mathcal{U}$. □

Corollary 3.1 *Let $\mathcal{S}=(\mathcal{U},\mathcal{G})$ be a $\mathcal{S}CAS$, $\mathcal{X}_1\subseteq\mathcal{U}$ and $\mathcal{X}_1$ is an exact set. Then, $\forall x\in\mathcal{U}$:*
(i) $\mu^{G}_{\mathcal{X}_1}(x,\mathcal{X}_1)=1\Leftrightarrow x\in\mathcal{X}_1$,
(ii) $\mu^{G}_{\mathcal{X}_1}(x,\mathcal{X}_1)=0\Leftrightarrow x\notin\mathcal{X}_1$.

Example 3.7 Suppose that an expert in the car industry wants to evaluate various car models. Let $U=\{c_1,c_2,c_3,c_4,c_5,c_6\}$ are the selected cars and $\mathcal{A}=\{a_1,a_2,a_3,a_4,a_5\}$ is a set of parameters related to the cars, such that a_1 refers to "performance", a_2 to "beautiful", a_3 to "luxurious", a_4 to "less fuel" and a_5 to "security". Consider $(U,\mathcal{G})$ is a soft rough covering approximation space $\mathcal{S}CAS$, where $G=(\mathcal{F},\mathcal{A})$ is soft set define in Table 2.

So, $\mathcal{M}N_S(c_1)=\{c_1,c_5\}$, $\mathcal{M}N_S(c_2)=\{c_2,c_5\}$, $\mathcal{M}N_S(c_3)=\{c_3,c_5\}$, $\mathcal{M}N_S(c_4)=\{c_4,c_5\}$, $\mathcal{M}N_S(c_5)=\{c_5\}$, $\mathcal{M}N_S(c_6)=\{c_2,c_3,c_5,c_6\}$. Suppose the set of "Excellent" cars is $\mathcal{X}_1=\{c_2,c_5\}$, hence $\underline{\mathcal{M}S}(\mathcal{X}_1)=\{c_2,c_5\}$ and $\overline{\mathcal{M}S}(\mathcal{X}_1)=\{c_2,c_3,c_4,c_5,c_6\}$. Then the accuracy measure is

$$\eta_{SC}(\mathcal{X}_1)=\frac{|\{c_2,c_5\}|}{|\{c_2,c_3,c_4,c_5,c_6\}|}=\frac{2}{5}$$

From the previous example, we deduce that our $\mathcal{S}RC$-model can be used for industrial purposes. Our model helps the experts in the evaluation of car's models and the expert can measure the accuracy of his evaluation. We believe that application will be helpful for the experts in process of car's industry and will support their evaluation process.

Table 2 $(G=\mathcal{F},\mathcal{A})$

$\mathcal{U}$	a_1	a_2	a_3	a_4	a_5	Result
c_1	1	0	1	1	0	*Good*
c_2	1	1	0	1	1	*Excellent*
c_3	0	1	1	0	1	*Good*
c_4	1	0	1	0	1	*Good*
c_5	1	1	1	1	1	*Excellent*
c_6	0	1	0	0	1	*Poor*

4 An attribute reduction via soft rough based on covering

Throughout this section, we introduce an example as an application for our approach. We introduce an algorithm for reduction of the attributes of the information systems via $\mathcal{S}CAS$. An attribute reduction supports the process of decision making.

Example 4.1 Suppose that $\mathcal{U} = \{p_2, p_1, p_4, p_3, p_5, p_6, p_7, p_8, p_8, p_9, p_{10}\}$ is a set of pilots. They are trained with respect to five attributes $\mathcal{A} = \{a_1, a_2, a_3, a_4, a_5\}$. An expert had evaluated them to determine whether they are sufficiently well trained with respect to these attributes or not. The results of evaluation are shown in the following information system Table 3.

Then $\mathcal{F}(a_1) = \{p_1, p_5, p_4, p_{10}, p_8\})$, $\mathcal{F}(a_2) = \{p_3, p_2, p_9, p_7\})$, $\mathcal{F}(a_3) = \{p_3, p_2, p_7, p_4, p_9\})$, $\mathcal{F}(a_4) = \{p_1, p_6, p_5, p_8\})$ and $\mathcal{F}(a_5) = \{p_3, p_2, p_6, p_5, p_9, p_{10}\})$. Consider the set of accepted pilots $\mathcal{X}_1 = \{p_4, p_3, p_7, p_9\}$. So, $\underline{\mathcal{M}S}(\mathcal{X}_1) = \{p_4\}$, $\overline{\mathcal{M}S}(\mathcal{X}_1) = \{p_3, p_2, P_4, p_9, p_7\}$ and $BND_{SC}(\mathcal{X}_1) = \{p_3, p_2, p_7, p_9\}$. We remove an attribute for each following case, hence the approximations operators are calculated as shown in the following Table 4.

Case 1: If the attribute a_1 is removed, then $\underline{\mathcal{M}S}(\mathcal{X}_1) = \emptyset$, $\overline{\mathcal{M}S}(\mathcal{X}_1) = \{p_3, p_2, p_4, p_7, p_9\}$, $BND_{SC}^{\mathcal{A}-a_1}(\mathcal{X}_1) = \{p_3, p_2, p_4, p_7, p_9\}$. Hence, $BND_{SC}(\mathcal{X}_1) \neq BND_{SC}^{\mathcal{A}-a_1}(\mathcal{X}_1)$.

Case 2: If the attribute a_2 is removed, then $\underline{\mathcal{M}S}(\mathcal{X}_1) = \{p_4\}$, $\overline{\mathcal{M}S}(\mathcal{X}_1) = \{p_3, p_2, p_4, p_7, p_9\}$, $BND_{SC}^{\mathcal{A}-a_2}(\mathcal{X}_1) = \{p_3, p_2, p_7, p_9\}$. Then $BND_{SC}(\mathcal{X}_1) = BND_{SC}^{\mathcal{A}-a_2}(\mathcal{X}_1)$ and a_2 is a superfluous attribute.

Case 3: If the attribute a_3 is removed, then $\underline{\mathcal{M}S}(\mathcal{X}_1) = \emptyset$, $\overline{\mathcal{M}S}(\mathcal{X}_1) = \{p_3, p_2, p_4, p_7, p_9\}$, $BND_{SC}^{\mathcal{A}-a_3}(\mathcal{X}_1) = \{p_3, p_2, p_4, p_7, p_9\}$. Then $BND_{SC}(\mathcal{X}_1) \neq BND_{SC}^{\mathcal{A}-a_3}(\mathcal{X}_1)$.

Case 4: If the attribute a_4 is removed, then $\underline{\mathcal{M}S}(\mathcal{X}_1) = \{p_4\}$, $\overline{\mathcal{M}S}(\mathcal{X}_1) = \{p_3, p_2, p_4, p_7, p_6, p_9, p_8\}$, $BND_{SC}^{\mathcal{A}-a_4}(\mathcal{X}_1) = \{p_3, p_2, p_7, p_6, p_9, p_8\}$. Then $BND_{SC}(\mathcal{X}_1) \neq BND_{SC}^{\mathcal{A}-a_4}(\mathcal{X}_1)$.

Case 5: If the attribute a_5 is removed, then $\underline{\mathcal{M}S}(\mathcal{X}_1) = \{p_4\}$, $\overline{\mathcal{M}S}(\mathcal{X}_1) = \{p_3, p_2, p_4, p_9, p_7, p_{10}\}$, $BND_{SC}^{\mathcal{A}-a_5}(\mathcal{X}_1) = \{p_3, p_2, p_9, p_7, p_{10}\}$. Then $BND_{SC}(\mathcal{X}_1) \neq BND_{SC}^{\mathcal{A}-a_5}(\mathcal{X}_1)$.

Table 3 Evaluation a set of pilots

$\mathcal{U}$	a_1	a_2	a_3	a_4	a_5	*Decision*
p_1	1	0	0	1	1	*Reject*
p_2	0	1	1	0	1	*Reject*
p_3	0	1	1	0	1	*Accept*
p_4	1	0	1	0	0	*Accept*
p_5	1	0	0	1	1	*Reject*
p_6	0	0	0	1	1	*Reject*
p_7	0	1	1	0	0	*Accept*
p_8	1	0	0	1	0	*Reject*
p_9	0	1	1	0	1	*Accept*
p_{10}	1	0	0	0	1	*Reject*

Table 4 Algorithm for reduction of attributes using soft rough based on covering

Algorithm	An attribute reduction using soft rough based on covering
Step 1	Input $\mathcal{S} = (\mathcal{U}, \mathcal{G})$, $\mathcal{G} = (\mathcal{F}, \mathcal{A})$, $\mathcal{A}$ is a set of attributes which symbolize table's information
Step 2	Calculate $\underline{\mathcal{M}S}(\mathcal{X}_1)$, $\overline{\mathcal{M}S}(\mathcal{X}_1)$, $BND_{SC}(\mathcal{X}_1)$ for the of accepted pilots
Step 3	Remove an attribute a_i from the set $\mathcal{A}$, and generate $\underline{\mathcal{M}S}(\mathcal{X}_1)$, $\overline{\mathcal{M}S}(\mathcal{X}_1)$, $BND_{SC}(\mathcal{X}_1)$ using $A - \{a_i\}$
Step 4	Reiterate step 3 for all attributes of $\mathcal{A}$
Step 5	If $\underline{\mathcal{M}S}(\mathcal{X}_1)$, $\overline{\mathcal{M}S}(\mathcal{X}_1)$, $BND_{SC}(\mathcal{X}_1)$ are equal for step 2 and step 3, then the attribute a_i is superfluous and is not important in decision making

From Algorithm 4, we deduce that the attribute a_2 is superfluous attribute and is not essential in the decision making of the accepted pilots. Hence reduct of attributes $\mathcal{A}$ is denoted by redcut $\mathcal{A} = \{a_1, a_3, a_4, a_5\}$.

5 Conclusion

The utility of the uncertainty theories is how to make a decision in problems with ambiguity or missing information. We explain fuzzy set theory $\mathcal{F}ST$, rough set theory $\mathcal{R}ST$ and soft set theory $\mathcal{S}ST$ and their generalizations among the theories of uncertainty. In our paper, we have developed a new of $\mathcal{S}RC$-model combining $\mathcal{S}ST$ and $\mathcal{R}ST$-based on covering. Compression between our model and existing models is discussed. We put forth an application for our model that can be helpful in the process of decision making. In future work, we will set forth other combinations among $\mathcal{F}ST$, $\mathcal{R}ST$ and $\mathcal{S}ST$ which are important for the problems of lack of information and fuzziness.

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